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On the inviscid stability of the laminar mixing of two parallel streams of a compressible fluid

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The inviscid stability of the laminar mixing of two parallel streams of a compressible fluid is investigated with respect to three-dimensional wavy disturbances. The flow is more unstable as the angle between the disturbance-wave-number vector and the principal-flow direction becomes larger. With three-dimensional disturbances, subsonic disturbances exist even at very high Mach number, and the flow is still unstable. However, it is also found that increasing the Mach number of the flow tends to stabilize the flow.

1. Introduction

The stability of the viscous flow that has no solid boundary was first investigated by Lessen (1949) and Chiarulli (1949) for the half jet, the laminar mixing of two parallel streams, of an incompressible fluid. Since then, many investigations on flows of this type have been made; among them the stability of a two-dimensional laminar jet of both incompressible and compressible fluid was explored by Pai (1951), that of the half jet of a compressible fluid was treated by Lin (1953), and some more cases were treated by Lessen, Fox and others (1954). However, they only considered two-dimensional wavy disturbances propagating along the main-flow direction. The stability of the jet-type flow of a compressible fluid with three-dimensional disturbances has rarely been treated.

For incompressible parallel flow, every three-dimensional disturbance was proved (Squire 1933) to be equivalent to a two-dimensional one at a lower Reynolds number; therefore, a consideration of two-dimensional disturbance was sufficient. For compressible flow, because of the complication of the disturbance equations, no prediction about which kind of disturbance tends to be more unstable was made until Dunn & Lin (1955) succeeded in simplifying the disturbance equations by an order-of-magnitude analysis. Contrary to the incompressible case, three-dimensional disturbances were found to play an important role in the stability problem of compressible flow. Recently, Lessen, Fox & Zien (1965) considered the instability of jets and wakes of an inviscid compressible fluid. It was shown that these flows are also more unstable as the wave-propagation angle relative to the main flow becomes larger.

The main purpose of the present paper is to investigate the influence of the Mach number of the flow and the angle of wave propagation on the stability characteristics of the laminar mixing of two streams of a compressible fluid.

Studies will be made in the case of infinite Reynolds number, not only because of its mathematical simplification but also because a great deal of information is given by this approach. We consider the flow with infinitesimal subsonic disturbances, i.e. the wave speed of the disturbances relative to the velocity of the flow in the direction of wave propagation is less than the local sonic speed; this is done because supersonic disturbances, which are often neglected in stability considerations, are less destabilizing than subsonic ones (Lessen *et al.* 1965).

To carry out the numerical calculation, we first need the velocity and the temperature profiles. Since it has been pointed out that the exact distributions of velocity and temperature have only a secondary influence on the stability characteristics in the mixing zone (Lin 1953), we use the approximations that the Prandtl number for the fluid is unity and that the viscosity varies linearly with the absolute temperature. With these approximations, we may apply the Howarth–Dorodnitzn transformation to obtain a new co-ordinate in which velocity distributions for different Mach numbers remain the same. The foregoing greatly simplifies the numerical calculations, and the final result is expected to be quite accurate even at higher Mach numbers.

2. Basic equations

Consider a two-dimensional flow of two parallel, semi-infinitely extended streams. Assume that the main flow is parallel and that it is subjected to small disturbances. Thus, with the initial upper stream velocity U_1^* as the reference velocity, the dimensionless velocity components in Cartesian co-ordinates are given by

$$u_x = \bar{u} + u', \quad u_y = v', \quad u_z = w', \tag{1}$$

and all other quantities by

$$q = \bar{q} + q'. \tag{2}$$

The main-flow pressure is assumed constant through the flow. All other main-flow quantities are functions of y only; all disturbance quantities are functions of x, y, z and t. The reference length is chosen as $l^* = (\nu_1^* x^*/U_1^*)^{\frac{1}{2}}$, and the reference time $t^* = l^*/U_1^*$, where ν_1^* is the kinetic viscosity of the initial upper stream, and quantities with asterisk superscripts are dimensional.

Consider the disturbance to be an oblique plane wave propagating at an angle with respect to the x-direction. The disturbance quantities in dimensionless form can be expressed as

$$\begin{aligned} u', v', w' &= \{f(y), \alpha\phi(y), h(y)\} \exp{[i(\alpha x + \beta z - \alpha c t)]}, \\ p', T', \rho' &= \{\pi(y), \theta(y), r(y)\} \exp{[i(\alpha x + \beta z - \alpha c t)]}, \end{aligned} \tag{3}$$

where p', T' and ρ' are the pressure, temperature and density disturbances to the flow respectively, with the initial upper-stream mean-flow quantities as reference quantities. The wave-propagation angle is obtained from the relation

$$\Theta = \cos^{-1}[\alpha(\alpha^2 + \beta^2)^{-\frac{1}{2}}]. \tag{4}$$

In (3), c is complex, that is $c = c_r + ic_i$. The real part of c is the wave-propagation velocity in the x-direction; the imaginary part of c indicates whether the disturbance is amplified, neutral, or damped, according to whether c_i is positive, zero, or negative.

For the case of infinite Reynolds number, the linearized disturbance equations for a compressible, heat-conducting fluid with constant specific heats are given by (Dunn & Lin 1955):

continuity, $$i\alpha(\bar{u}-c)\,r+\alpha\bar{\rho}'\phi+\bar{\rho}(i\alpha f+i\beta h+\alpha\phi')=0; \tag{5}$$

momentum, $$\bar{\rho}[i(\bar{u}-c)f+\bar{u}'\phi]=-(i/\gamma M_0^2)\,\pi, \tag{6}$$

$$i\alpha^2\bar{\rho}(\bar{u}-c)\,\phi=-(1/\gamma M_0^2)\,\pi', \tag{7}$$

$$\alpha\bar{\rho}(\bar{u}-c)\,h=-(\beta/\gamma M_0^2)\,\pi; \tag{8}$$

energy, $$\alpha\bar{\rho}[i(\bar{u}-c)\,\theta+\bar{T}'\phi]=-(\gamma-1)\,\bar{p}\bar{T}(i\alpha f+i\beta h+\alpha\phi'); \tag{9}$$

state, $$\pi/\bar{p}=r/\bar{\rho}+\theta/\bar{T}; \tag{10}$$

here γ is the specific heats ratio, M_0 the Mach number of the upper stream, and the primes represent the derivatives of a quantity with respect to y. In the above equations the Prandtl number is taken as unity, and the effect of gravity is neglected.

Here we introduce the following transformations which are due to Squire (1933)

$$\left.\begin{array}{ll}\tilde{\alpha}\tilde{f}=\alpha f+\beta h, & \tilde{\alpha}\tilde{\theta}=\alpha\theta,\\ \tilde{\alpha}\tilde{\phi}=\alpha\phi, & \tilde{\alpha}\tilde{m}=\alpha m,\\ \tilde{\alpha}\tilde{\pi}=\alpha\pi, & \tilde{\alpha}=\sqrt{(\alpha^2+\beta^2)},\\ \tilde{\alpha}\tilde{r}=\alpha r, & \tilde{\alpha}\tilde{M}=\alpha M_0.\end{array}\right\} \tag{11}$$

Thus, equations (5) to (10) become

$$i(\bar{u}-c)\,\tilde{r}+\bar{\rho}(\tilde{\phi}'+i\tilde{f})+\bar{\rho}'\tilde{\phi}=0, \tag{12}$$

$$\bar{\rho}[i(\bar{u}-c)\tilde{f}+\bar{u}'\tilde{\phi}]=-i\tilde{\pi}/\gamma\tilde{M}^2, \tag{13}$$

$$\tilde{\alpha}^2\bar{\rho}[i(\bar{u}-c)\,\tilde{\phi}]=-\tilde{\pi}'/\gamma\tilde{M}^2, \tag{14}$$

$$\bar{\rho}[i(\bar{u}-c)\,\tilde{\theta}+\bar{T}'\tilde{\phi}]=-(\gamma-1)\,\bar{p}\bar{T}(\tilde{\phi}+i\tilde{f}), \tag{15}$$

$$\tilde{\pi}/\bar{p}=\tilde{r}/\bar{\rho}+\tilde{\theta}/\bar{T}. \tag{16}$$

These transformed equations have the same form as those for the flow with two-dimensional disturbances. They can be reduced into a differential equation for each dependent variable. For the transformed pressure disturbance, we have

$$\tilde{\pi}''-\left(\frac{2\bar{u}'}{\bar{u}-c}-\frac{\bar{T}'}{\bar{T}}\right)\tilde{\pi}'-\tilde{\alpha}^2\left[1-\frac{\tilde{M}^2}{\bar{T}}(\bar{u}-c)^2\right]\tilde{\pi}=0. \tag{17}$$

3. Velocity and temperature distributions

To obtain an approximate velocity profile for stability considerations, we will assume that the viscosity varies linearly with the absolute temperature (Lees 1947). With this assumption, the steady-state equation of motion and the energy equation become uncoupled. We will follow the procedure given by Howarth (1948) to obtain a new co-ordinate system (x^*, Y^*) in which the velocity distributions remain the same for different Mach numbers.

Let
$$Y^*=\int_0^{y^*}\frac{\bar{T}_0^*}{\bar{T}^*}\,dy^*,\dagger \tag{18}$$

† In this section, all quantities are dimensional.

and the stream function

$$\psi^*(x^*, y^*) = \chi^*(x^*, Y^*). \tag{19}$$

From the continuity equation, the velocity is related to the stream function by

$$\bar{u}^* = \frac{\bar{\rho}_0^*}{\bar{\rho}^*}\left(\frac{\partial \psi^*}{\partial y^*}\right)_{x^*} = \frac{\bar{\rho}_0^*}{\bar{\rho}^*}\left(\frac{\partial \chi^*}{\partial Y^*}\right)_{x^*}\left(\frac{\partial Y^*}{\partial y^*}\right)_{x^*}. \tag{20}$$

Since the pressure is assumed constant, one may write for an ideal gas

$$\bar{\rho}^*/\bar{\rho}_0^* = \bar{T}_0^*/\bar{T}^*. \tag{21}$$

Therefore,

$$\bar{u}^* = \partial \chi^*/\partial Y^*. \tag{22}$$

The equation of motion for the steady flow of a compressible fluid is

$$\bar{u}^* \frac{\partial \bar{u}^*}{\partial x^*} = \frac{1}{\bar{\rho}^*}\frac{\partial}{\partial y^*}\left(\bar{\mu}^* \frac{\partial u^*}{\partial y^*}\right). \tag{23}$$

The derivatives of the velocity are then

$$\frac{\partial \bar{u}^*}{\partial x^*} = \left(\frac{\partial \bar{u}^*}{\partial x^*}\right)_{Y^*} + \left(\frac{\partial \bar{u}^*}{\partial Y^*}\right)_{x^*}\left(\frac{\partial Y^*}{\partial x^*}\right)_{y^*} = \frac{\partial^2 \chi^*}{\partial x^* \partial Y^*} + \frac{\partial^2 \chi^*}{\partial Y^{*2}}\left(\frac{\partial Y^*}{\partial x^*}\right)_{y^*} \tag{24}$$

and

$$\frac{\partial \bar{u}^*}{\partial y^*} = \frac{\bar{T}_0^*}{\bar{T}^*}\frac{\partial^2 \chi^*}{\partial Y^{*2}}. \tag{25}$$

By introducing the relation

$$\bar{\mu}^*/\bar{\mu}_0^* = \bar{T}^*/\bar{T}_0^*, \tag{26}$$

equation (23) becomes, in (x^*, Y^*) co-ordinates,

$$\frac{\partial^2 \chi^*}{\partial x^* \partial Y^*}\frac{\partial \chi^*}{\partial Y^*} - \frac{\partial^2 \chi^*}{\partial Y^{*2}}\frac{\partial \chi^*}{\partial x^*} = \frac{\bar{\mu}_0^*}{\bar{\rho}_0^*}\frac{\partial^3 \chi^*}{\partial Y^{*3}}. \tag{27}$$

Since for an incompressible fluid, the equation of motion in terms of stream function is given by

$$\frac{\partial^2 \psi^*}{\partial x^* \partial y^*}\frac{\partial \psi^*}{\partial y^*} - \frac{\partial^2 \psi^*}{\partial y^{*2}}\frac{\partial \psi^*}{\partial x^*} = \frac{\bar{\mu}_0^*}{\bar{\rho}_0^*}\frac{\partial^3 \psi^*}{\partial y^{*3}}, \tag{28}$$

we see that, regardless of different Mach numbers, the velocity distribution in (x^*, Y^*) co-ordinates in our approximation is the same as that in (x^*, y^*) co-ordinates for an incompressible fluid.

We further assume that the flow is iso-energetic, so that

$$C_p^* \bar{T}^* + \tfrac{1}{2}\bar{\rho}^* \bar{u}^{*2} = \text{const.} \tag{29}$$

Therefore, the temperature is given by

$$\bar{T}^* = \bar{T}_0^*[1 + \tfrac{1}{2}(\gamma - 1) M_0^2\{1 - (\bar{u}^{*2}/\bar{u}_0^{*2})\}]. \tag{30}$$

4. Inviscid solution

In a dimensionless Y-co-ordinate, equation (17) is written as

$$\ddot{\tilde{\pi}} - [2\dot{\bar{u}}/(\bar{u} - c)]\dot{\tilde{\pi}} - \tilde{\alpha}^2 \bar{T}[\bar{T} - \tilde{M}^2(\bar{u} - c)^2]\tilde{\pi} = 0, \tag{31}$$

where

$$(\dot{\ }) = d(\ )/dY.$$

The asymptotic behaviour of $\tilde{\pi}$ is that

$$\left.\begin{aligned} \tilde{\pi} &\sim \exp\left[-\tilde{\alpha}\{1-\tilde{M}^2(1-c)^2\}^{\frac{1}{2}}\,Y\right] \quad \text{as} \quad Y \to +\infty, \\ \tilde{\pi} &\sim \exp\left[\tilde{\alpha}\overline{T}(-\infty)\left\{1-\frac{\tilde{M}^2}{\overline{T}(-\infty)}(\overline{u}(-\infty)-c)^2\right\}^{\frac{1}{2}}Y\right] \quad \text{as} \quad Y \to -\infty. \end{aligned}\right\} \tag{32}$$

Let
$$G = \dot{\tilde{\pi}}/\tilde{\alpha}^2\overline{T}\tilde{\pi}. \tag{33}$$

Then equation (31) becomes

$$\dot{G} = [\overline{T}-\tilde{M}^2(\overline{u}-c)^2] + \left(\frac{2\dot{\overline{u}}}{\overline{u}-c}-\frac{\dot{\overline{T}}}{\overline{T}}\right)G - \tilde{\alpha}^2\overline{T}G^2. \tag{34}$$

From (32), the boundary conditions for G are

$$G(\pm\infty) = \mp\frac{1}{\tilde{\alpha}}\left\{1-\frac{\tilde{M}^2}{\overline{T}(\pm\infty)}[\overline{u}(\pm\infty)-c]^2\right\}^{\frac{1}{2}}. \tag{35}$$

For a neutral inviscid disturbance Lees & Lin (1946) have shown that the necessary and sufficient condition for its existence is that

$$\left[\frac{d}{dy}\left(\frac{1}{\overline{T}}\frac{d\overline{u}}{dy}\right)\right]_c = 0 \tag{36}$$

at the critical point. From this equation we can determine the neutral wave speed c_s. Thus, our problem reduces to the determination of the characteristic value $\tilde{\alpha}$ that satisfies (34) and (35).

Note that there is a singularity at $\overline{u} = c_s$. To avoid the singularity we integrate (34) along the path below the singularity in the complex Y-plane. By choosing this path, G, $\overline{u}$, and $\overline{T}$ in (34) are complex, but they must have real values on the real axis for neutral disturbances. It is also possible to determine the characteristic values for amplified or damped disturbances by integration along this path.

5. Partial derivatives of c

Some important stability characteristics can be examined from the changes of c_i with respect to $\tilde{\alpha}$, M_0 and Θ. It has been shown that $[\partial c/\partial(-\alpha^2)]_{M_0,\Theta}$ always has a positive imaginary part (Lin 1953). Thus, for any wave-number α larger than the neutral wave-number the flow is stable, at least for large Reynolds numbers. Here, we will derive expressions for

$$(\partial c/\partial\tilde{\alpha})_{M_0,\Theta}, \quad (\partial c/\partial\Theta)_{M_0,\tilde{\alpha}} \quad \text{and} \quad (\partial c/\partial M_0)_{\Theta\,\tilde{\alpha}}.$$

The differentiation of equation (31) with respect to $\tilde{\alpha}$, keeping M_0 and Θ constant, yields

$$\begin{aligned} \ddot{\tilde{\pi}}_{\tilde{\alpha}} - \frac{2\dot{\overline{u}}}{\overline{u}-c}\dot{\tilde{\pi}}_{\tilde{\alpha}} - \frac{2\dot{\overline{u}}}{(\overline{u}-c)^2}\dot{\tilde{\pi}}\left(\frac{\partial c}{\partial\tilde{\alpha}}\right)_{M_0,\Theta} - \tilde{\alpha}^2\overline{T}[\overline{T}-\tilde{M}^2(\overline{u}-c)^2]\,\tilde{\pi}_{\tilde{\alpha}} \\ - 2\tilde{\alpha}\overline{T}[\overline{T}-\tilde{M}^2(\overline{u}-c)^2]\,\tilde{\pi} - 2\tilde{\alpha}^2\overline{T}\tilde{M}^2(\overline{u}-c)\,\tilde{\pi}(\partial c/\partial\tilde{\alpha})_{M_0,\Theta} = 0, \end{aligned} \tag{37}$$

where
$$(\;)_{\tilde{\alpha}} = [\partial(\;)/\partial\tilde{\alpha}]_{M_0,\Theta}.$$

Multiplying (37) by $\tilde{\pi}$ and (31) by $\tilde{\pi}_{\tilde{\alpha}}$, and subtracting the results, one gets

$$\ddot{\tilde{\pi}}\tilde{\pi}_{\tilde{\alpha}}-\ddot{\tilde{\pi}}_{\tilde{\alpha}}\tilde{\pi}-[2\dot{\bar{u}}/(\bar{u}-c)](\dot{\tilde{\pi}}\tilde{\pi}_{\tilde{\alpha}}-\dot{\tilde{\pi}}_{\tilde{\alpha}}\tilde{\pi})$$
$$=-\frac{2\dot{\bar{u}}}{(\bar{u}-c)^2}\dot{\tilde{\pi}}\tilde{\pi}\left(\frac{\partial c}{\partial\tilde{\alpha}}\right)_{M_0,\Theta}-2\tilde{\alpha}^2\bar{T}\tilde{M}^2(\bar{u}-c)\tilde{\pi}^2\left(\frac{\partial c}{\partial\tilde{\alpha}}\right)_{M_0,\Theta}-2\tilde{\alpha}\bar{T}[\bar{T}-\tilde{M}^2(\bar{u}-c)^2]\tilde{\pi}^2. \tag{38}$$

Multiplying equation (38) by $1/(\bar{u}-c)^2$, integrating from $Y=-\infty$ to $Y=+\infty$, and noting that

$$\int_{-\infty}^{+\infty}\left[\frac{\ddot{\tilde{\pi}}\tilde{\pi}_{\tilde{\alpha}}-\tilde{\pi}\ddot{\tilde{\pi}}_{\tilde{\alpha}}}{(\bar{u}-c)^2}-2\frac{\dot{\tilde{\pi}}\tilde{\pi}_{\tilde{\alpha}}-\tilde{\pi}\dot{\tilde{\pi}}_{\tilde{\alpha}}}{(\bar{u}-c)^3}\dot{\bar{u}}\right]dY=\left.\frac{\tilde{\pi}_{\tilde{\alpha}}\dot{\tilde{\pi}}-\tilde{\pi}\dot{\tilde{\pi}}_{\tilde{\alpha}}}{(\bar{u}-c)^2}\right|_{Y=-\infty}^{Y=+\infty}=0, \tag{39}$$

we obtain
$$\left(\frac{\partial c}{\partial\tilde{\alpha}}\right)_{M_0,\Theta}=-\frac{1}{S_i}\int_{-\infty}^{+\infty}\tilde{\alpha}\bar{T}\left[\frac{\bar{T}}{(\bar{u}-c)^2}-\tilde{M}^2\right]\tilde{\pi}^2\,dY, \tag{40}$$

where
$$S_i=\int_{-\infty}^{+\infty}\left[\frac{\dot{\bar{u}}\dot{\tilde{\pi}}\tilde{\pi}}{(\bar{u}-c)^4}+\frac{\tilde{\alpha}^2\bar{T}\tilde{M}^2}{\bar{u}-c}\tilde{\pi}^2\right]dY.$$

By repeating the same procedure, but keeping only Θ constant, it follows that

$$\left(\frac{\partial c}{\partial\tilde{\alpha}}\right)_{\Theta}=\left(\frac{\partial c}{\partial\tilde{\alpha}}\right)_{\Theta,M_0}-\frac{1}{2S_i}\left(\frac{\partial M_0}{\partial\tilde{\alpha}}\right)_{\Theta}\int_{-\infty}^{+\infty}\tilde{\alpha}^2\tilde{\pi}^2\left\{\frac{\partial\bar{T}}{dM_0}\left[\frac{2\bar{T}}{(\bar{u}-c)^2}-\tilde{M}^2\right]-2\tilde{M}\bar{T}\cos\Theta\right\}dY. \tag{41}$$

In obtaining equation (41) we have taken advantage of the fact that the velocity is independent of the Mach number in the transformed Y-co-ordinate. Since

$$\left(\frac{\partial c}{\partial\tilde{\alpha}}\right)_{\Theta}=\left(\frac{\partial c}{\partial\tilde{\alpha}}\right)_{\Theta,M_0}+\left(\frac{\partial c}{\partial M_0}\right)_{\Theta,\tilde{\alpha}}\left(\frac{\partial M_0}{\partial\tilde{\alpha}}\right)_{\Theta}, \tag{42}$$

it follows from (41) that

$$\left(\frac{\partial c}{\partial M_0}\right)_{\Theta,\tilde{\alpha}}=-\frac{1}{2S_i}\int_{-\infty}^{+\infty}\tilde{\alpha}^2\tilde{\pi}^2\left\{\frac{d\bar{T}}{dM_0}\left[\frac{2\bar{T}}{(\bar{u}-c)^2}-\tilde{M}^2\right]-2\tilde{M}\bar{T}\cos\Theta\right\}dY. \tag{43}$$

The expression for $(\partial c/\partial\Theta)_{M_0,\tilde{\alpha}}$ is obtained in a similar way as

$$\left(\frac{\partial c}{\partial\Theta}\right)_{M_0,\tilde{\alpha}}=-\frac{1}{S_i}\sin\Theta\,\tilde{\alpha}^2\tilde{M}M_0\int_{-\infty}^{+\infty}\bar{T}\tilde{\pi}^2\,dY. \tag{44}$$

6. Numerical calculations

Numerical calculation has been made for the flow with initially zero velocity for the lower stream. The velocity distributions along the real axis of Y and along the straight line path $-6-3i$ to $6+0i$ in the complex Y-plane are calculated by using the method given by Lessen (1950). The calculation starts from the asymptotic solution at large negative Y

$$F(Y)=a\left(-1+e^{\frac{1}{2}aY}-\tfrac{1}{4}e^{aY}+\frac{1}{14\cdot 4}e^{\frac{3}{2}aY}+\dots\right), \tag{45}$$

where $a=1\cdot 23849387$. The function F satisfies the boundary-layer equation

$$F\ddot{F}+2\dddot{F}=0, \tag{46}$$

and $\bar{u}$ is obtained from $$\bar{u}(Y) = \dot{F}(Y). \tag{47}$$

Higher derivatives of F are easily obtained; values of F and its derivatives can then be obtained step by step along the desired paths by a Taylor series expansion. Table 1 gives the neutral wave speed for different Mach numbers.

M	c_s	M	c_s
0·0	0·5872700	2·0	0·7666133
0·5	0·6029153	3·0	0·8576864
1·0	0·6460949	4·0	0·9084932
1·5	0·7056756	5·0	0·9369957

TABLE 1. Neutral wave speed c_s for different Mach numbers

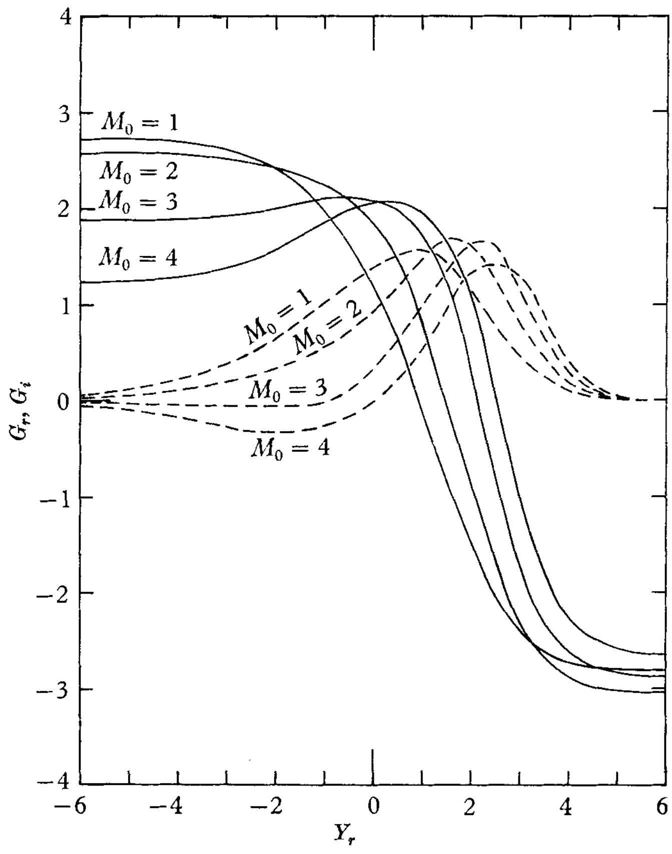

FIGURE 1. Function G (G_r solid lines and G_i broken lines) for different Mach numbers along the complex path $Y_i = \frac{1}{4}Y_r - 1{\cdot}5$ at $\Theta = 60°$.

The integration of (34) also starts from the asymptotic solution

$$G(Y) = b_0 + b_1 e^{\frac{1}{2}aY} + b_2 e^{aY} + b_3 e^{\frac{3}{2}aY} + \ldots, \tag{48}$$

where b_0, b_1, b_2 and b_3 are given in the Appendix. For illustration, values for G for different Mach numbers at $\Theta = 60°$ are plotted in figure 1.

At small Mach numbers, the pressure disturbance along the complex path, shown in figures 2, 3 and 4, has the same behaviour as the vertical velocity

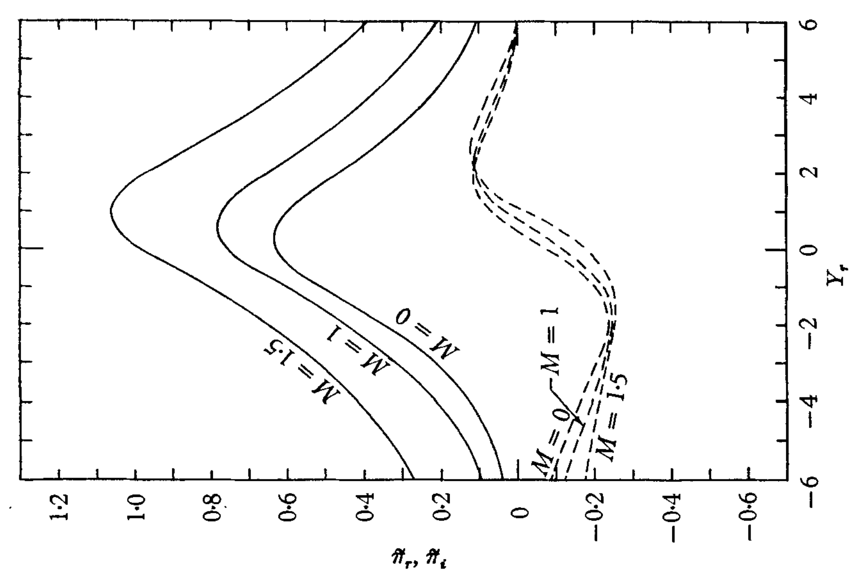

FIGURE 2. Pressure disturbance ϖ (ϖ_r solid lines and ϖ_i broken lines) along the complex path $Y_i = \frac{1}{4}Y_r - 1.5$ at $\Theta = 0°$.

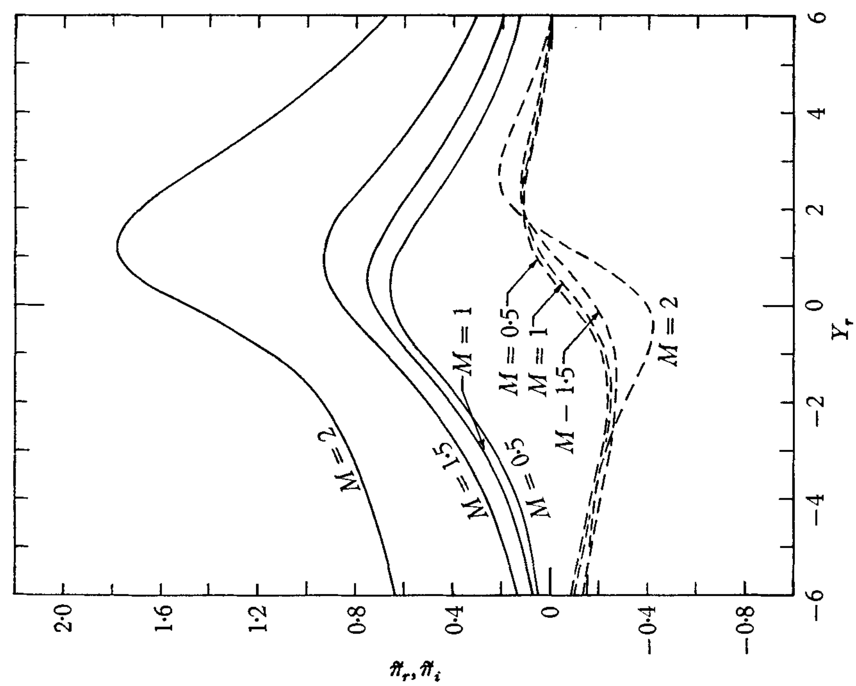

FIGURE 3. Pressure disturbance ϖ (ϖ_r solid lines and ϖ_i broken lines) along the complex path $Y_i = \frac{1}{4}Y_r - 1.5$ at $\Theta = 30°$.

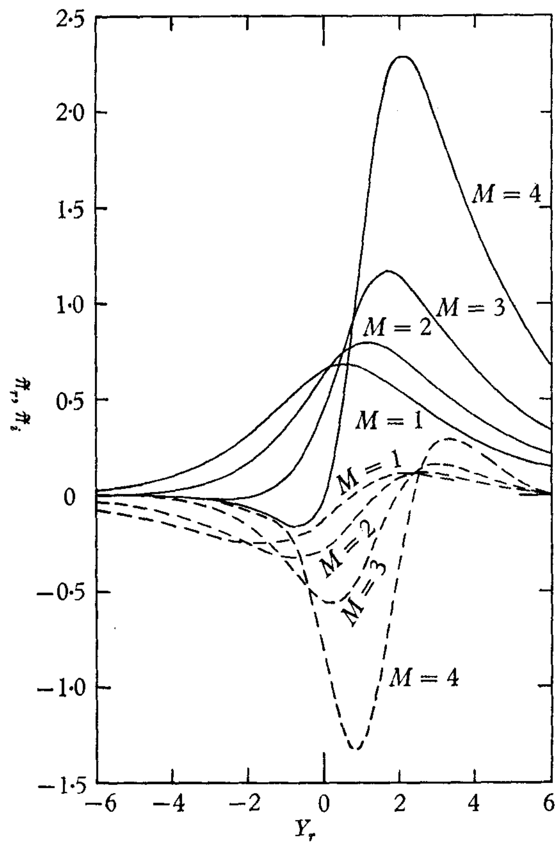

FIGURE 4. Pressure disturbance $\mathcal{P}$ ($\mathcal{P}_r$ solid lines and $\mathcal{P}_i$ broken lines) along the complex path $Y_i = \frac{1}{4}Y_r - 1{\cdot}5$ at $\Theta = 75°$.

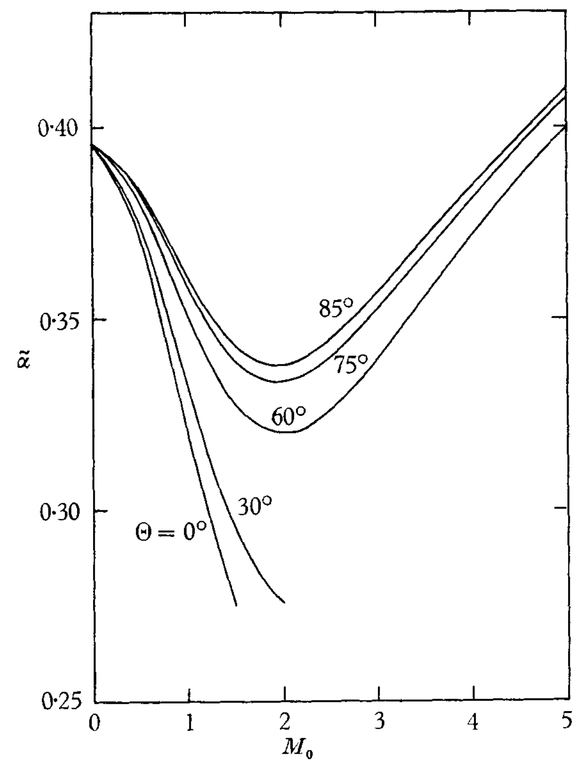

FIGURE 5. Neutral wave-number $\tilde{\alpha}$ *vs* Mach number M_0 for different wave-propagation angles Θ.

M_0 \ Θ	0°	30°	60°	75°	85°
0·0	0·2592 − 0·8170i	—	—	—	—
0·5	0·3140 − 0·8240i	0·3111 − 0·8231i	0·3056 − 0·8213i	0·3036 − 0·8207i	0·3029 − 0·8205i
1·0	0·4615 − 0·7833i	0·4465 − 0·7825i	0·4199 − 0·7839i	0·4111 − 0·7850i	0·4083 − 0·7854i
1·5	0·5745 − 0·6161i	0·5417 − 0·6232i	0·4966 − 0·6448i	0·4824 − 0·6536i	0·4780 − 0·6565i
2·0	—	0·5028 − 0·4316i	0·4616 − 0·4724i	0·4501 − 0·4863i	0·4464 − 0·4908i
3·0	—	—	0·2874 − 0·2530i	0·2832 − 0·2626i	0·2818 − 0·2656i
4·0	—	—	0·1732 − 0·1533i	0·1715 − 0·1578i	0·1709 − 0·1592i
5·0	—	—	0·1121 − 0·1018i	0·1113 − 0·1040i	0·1110 − 0·1047i

TABLE 2. $(\partial c/\partial \tilde{\alpha})_{M_0, \Theta}$ for different Mach numbers and wave-propagation angles

M_0 \ Θ	0°	30°	60°	75°	85°
0·0	0·0000 + 0·0000i	—	—	—	—
0·5	0·0000 + 0·0000i	−0·0031 + 0·0080i	−0·0030 + 0·0079i	−0·0017 + 0·0046i	−0·0006 + 0·0016i
1·0	0·0000 + 0·0000i	−0·0150 + 0·0254i	−0·0133 + 0·0245i	−0·0074 + 0·0140i	−0·0025 + 0·0048i
1·5	0·0000 + 0·0000i	−0·0313 + 0·0343i	−0·0255 + 0·0329i	−0·0139 + 0·0188i	−0·0047 + 0·0065i
2·0	—	−0·0746 + 0·0523i	−0·0431 + 0·0302i	−0·0152 + 0·0164i	−0·0052 + 0·0057i
3·0	—	—	−0·0164 + 0·0145i	−0·0089 + 0·0083i	−0·0031 + 0·0029i
4·0	—	—	−0·0078 + 0·0069i	−0·0043 + 0·0039i	−0·0015 + 0·0014i
5·0	—	—	−0·0039 + 0·0035i	−0·0022 + 0·0020i	−0·0008 + 0·0007i

TABLE 3. $(\partial c/\partial \Theta)_{M_0, \tilde{\alpha}}$ for different Mach numbers and wave-propagation angles

M_0 \ Θ	0°	30°	60°	75°	85°
0·0	0·0000 − 0·0000i	—	—	—	—
0·5	−0·0069 − 0·0674i	−0·0111 − 0·0574i	−0·0190 − 0·0378i	−0·0218 − 0·0308i	−0·0227 − 0·0285i
1·0	0·0192 − 0·1193i	0·0051 − 0·1008i	−0·0184 − 0·6591i	−0·0258 − 0·0537i	−0·0281 − 0·0499i
1·5	0·0677 − 0·1200i	0·0370 − 0·1016i	−0·0027 − 0·0672i	−0·0137 − 0·0555i	−0·0170 − 0·0518i
2·0	—	0·0583 − 0·0732i	0·0066 − 0·0487i	−0·0033 − 0·0407i	−0·0063 − 0·0382i
3·0	—	—	0·0049 − 0·0179i	0·0008 − 0·0151i	0·0004 − 0·0142i
4·0	—	—	0·0019 − 0·0066i	0·0005 − 0·0056i	0·0000 − 0·0052i
5·0	—	—	0·0008 − 0·0027i	0·0002 − 0·0023i	0·0000 − 0·0022i

TABLE 4. $(\partial c/\partial M_0)_{\Theta, \tilde{\alpha}}$ for different Mach numbers and wave-propagation angles

disturbance along the real path calculated by Lin (1953). At higher Mach numbers, $\tilde{\pi}$ has one or more nodes, and between $Y_r = 1$ to 4 for $M_0 > 4$ the slope is very steep, which is partly due to the co-ordinate transformation.

Figure 5 gives values of $\tilde{\alpha}$ corresponding to neutral subsonic disturbance for different Mach numbers at different wave-propagation angles. For incompressible flow, the value agrees with that obtained by Lessen (1950).

The contribution of integrations for $(\partial c/\partial\tilde{\alpha})_{M_0,\Theta}$, $(\partial c/\partial\Theta)_{M_0,\tilde{\alpha}}$ and $(\partial c/\partial M_0)_{\Theta,\tilde{\alpha}}$ from $Y = -\infty$ to $-6-3i$ and from $Y = 6+0i$ to $+\infty$ are small; therefore, integrations were carried out from $Y = -6-3i$ to $6+0i$. The results are shown in tables 2, 3 and 4.

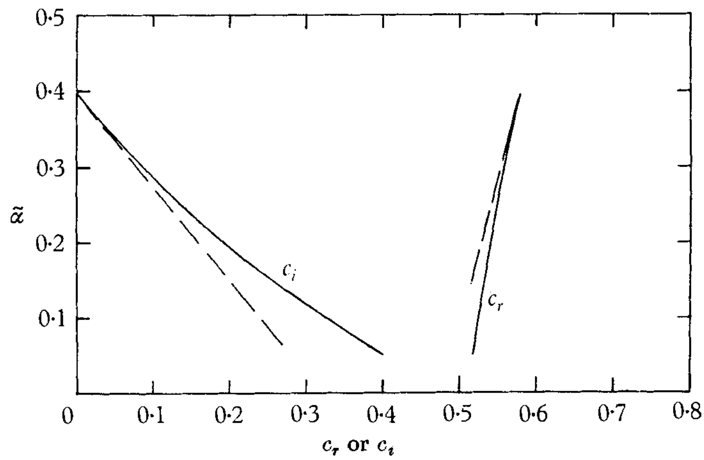

FIGURE 6. $\tilde{\alpha}$ *vs* c for $M = 0$. Curves are obtained by Lessen, Fox *et al.* (1954). Broken lines are slopes obtained from the present calculation.

In figure 6, we compare our result of $(\partial c/\partial\tilde{\alpha})_{M_0,\Theta}$ for the incompressible case with the (c, α) curve obtained by Lessen, Fox *et al.* (1954). Lin (1953) calculated $(\partial c/\partial\tilde{\alpha})_{M_0,\Theta}$ for the incompressible case and for $M_0 = 1$ at $\Theta = 0°$; he had the values of 0·093–0·287i and 0·177–0·209i, respectively. Since he took a reference length which is $2^{\frac{3}{2}}$ times that of ours, the results are in good agreement with our 0·259–0·817i and 0·462–0·783i.

A comparison of the present result with the result obtained from the stability of a single, plane vortex sheet has been made. For a plane vortex sheet the flow is unstable, for any wave-number at $\Theta = 0°$, when the Mach number is smaller than $2^{\frac{3}{2}}$ for the case of equal sonic velocities of both streams, or 2·5 for the case of iso-energetic flow (Miles 1958). While in our case, when $\Theta = 0°$, neutral stability can exist for certain wave-numbers even at zero and very small Mach number. However, general stability characteristics, such as the influences of the Mach number and the wave-propagation angle, are similar in both considerations.

7. Conclusions

As a result of the foregoing calculations for the laminar mixing of two compressible streams, we draw the following conclusions for the case of infinite or very large Reynolds number:

(1) The flow is more unstable as the wave-propagation angle Θ becomes larger, since the sign of the imaginary part of $(\partial c/\partial\Theta)_{M_0,\tilde{\alpha}}$ is always positive (table 3). However, the rate of change of c_i with respect to Θ decreases as Θ increases.

(2) The previous result (Lin 1953), that when $M_0 > 1{\cdot}7$ the flow is stable because of the non-existence of the subsonic disturbance, is only true for disturbances propagating along the flow direction. At a larger wave-propagation angle a subsonic disturbance can exist, and the flow continues to be unstable.

(3) As the Mach number increases the flow becomes less unstable. This is clear from table 4.

(4) The range of wave-number for instability has a minimum value at about $M_0 = 2$ (figure 5).

(5) The increase of wave-number at a given Mach number and Θ is always less destabilizing, because $(\partial c/\partial\tilde{\alpha})_{M_0,\Theta}$ always has a negative imaginary part (table 2).

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Appendix. Asymptotic solutions

At large negative Y, from (45) and (47), the velocity is given by

$$\bar{u} = a^2[\tfrac{1}{2}e^{\frac{1}{2}aY} - \tfrac{1}{4}e^{aY} + (3/28{\cdot}8)\,e^{\frac{3}{2}aY} + \ldots]. \tag{A 1}$$

From (30), the temperature has the form

$$\bar{T} = [1+\tfrac{1}{2}(\gamma-1)M_0^2] + \tfrac{1}{8}(\gamma-1)\,a^4M_0^2(e^{aY} - e^{\frac{3}{2}aY} + \ldots). \tag{A 2}$$

Thus (34) becomes

$$\dot{G} = (h_0 + h_1\,e^{\frac{1}{2}aY} + h_2\,e^{aY} + h_3\,e^{\frac{3}{2}aY} + \ldots) + (h_4\,e^{\frac{1}{2}aY} + h_5\,e^{aY} + h_6\,e^{\frac{3}{2}aY})\,G + (h_7 + h_8\,e^{aY} + h_9\,e^{\frac{3}{2}aY})\,G^2, \tag{A 3}$$

where

$$h_0 = 1+\tfrac{1}{2}(\gamma-1)M_0^2 - \tilde{M}^2c^2,$$
$$h_1 = \tilde{M}^2ca^2,$$
$$h_2 = -\tfrac{1}{2}[\tilde{M}^2(ca^2+\tfrac{1}{2}a^4) + \tfrac{1}{4}(\gamma-1)M_0^2\,a^4],$$
$$h_3 = a^4[\tfrac{1}{8}(\gamma-1)M_0^2 + \tfrac{1}{4}\tilde{M}^2] + (3/14{\cdot}4)\tilde{M}^2ca^2,$$
$$h_4 = -a^3/2c,$$
$$h_5 = -\frac{a^3}{2c}\left(\frac{a^2}{2c}-1\right) + \frac{(\gamma-1)\,M_0^2\,a^5}{8[1+\frac{1}{2}(\gamma-1)M_0^2]},$$
$$h_6 = -\frac{3}{16}\frac{(\gamma-1)\,M_0^2\,a^5}{1+\frac{1}{2}(\gamma-1)M_0^2} + \frac{a^3}{2c}\left[\frac{a^2}{2c}\left(\frac{a^2}{2c}-\frac{3}{2}\right) + \frac{9}{14{\cdot}4}\right],$$
$$h_7 = -\tilde{\alpha}^2[1+\tfrac{1}{2}(\gamma-1)M_0^2],$$
$$h_8 = \tfrac{1}{8}\tilde{\alpha}^2a^4(\gamma-1)\,M_0^2,$$

and

$$h_9 = -h_8.$$

Equation (A 3) implies the following solution for G,

$$G = b_0 + b_1\,e^{\frac{1}{2}aY} + b_2\,e^{aY} + b_3\,e^{\frac{3}{2}aY} + \ldots. \tag{A 4}$$

The constants are given by

$$b_0 = (-h_0/h_7)^{\frac{1}{2}},$$
$$b_1 = (h_1 + b_0 h_4)/(\tfrac{1}{2}a - 2b_0 h_7),$$
$$b_2 = (h_2 + b_1 h_4 + b_0 h_5 + b_1^2 h_7 + b_0^2 h_8)/(a - 2b_0 h_7),$$

and $$b_3 = (h_3 + b_0[h_6 + h_8(2h_0 - b_0)] + b_1(h_1 b_2 + h_5) + b_2 h_4)/(3a/2 - 2b_0 h_7).$$

The asympotic solution for $\tilde{\pi}$ can be written as

$$\tilde{\pi}(Y) = \pi_0(Y) + \pi_1(Y)\, e^{\frac{1}{2}aY} + \pi_2(Y)\, e^{aY} + \dots \qquad \text{(A 5)}$$

Substitute this into equation (33), equate terms of the same power of e^{aY} and one obtains

$$\pi_0 = e^{b_0 sY},$$
$$\pi_1 = (2/a)\, s b_1 e^{b_0 sY},$$

and $$\pi_2 = (\tilde{\alpha}^2/a)[-\tfrac{1}{8}(\gamma-1)M_0^2 a^4 b_0 + \{1 + \tfrac{1}{2}(\gamma-1)M_0^2\}(b_2 + (2b_1^2/a)\, s)]\, e^{b_0 sY},$$

where $$s = \tilde{\alpha}^2[1 + \tfrac{1}{2}\gamma(-1)M_0^2].$$

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